Ex. 13.6

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A set of N basis functions $\varphi_n(x)$, n = 1,2,…,N, is defined on the interval $0 \le x \le L$. Given the velocity $U(x)$, the filtered velocity $\bar{U}(x)$ is defined as the least-square projection of $U(x)$ onto the basis functions. That is, $\bar{U}(x)$ is given by

$$\bar{U}(x) = \sum_{n=1}^{N} a_n \varphi_n(x) \tag{1}$$

where the basis-function coefficients a_n are determined by the condition that they minimize the mean-square residual

$$\chi \equiv \frac{1}{L}\int_0^L \left[\bar{U}(x) - U(x)\right]^2 \mathrm{d}x \tag{2}$$

By substituting Eq. (1) into Eq. (2) and differentiating with respect to a_m, show that the coefficients satisfy the matrix equation

$$\sum_{n=1}^{N} B_{mn} a_n = v_m \tag{3}$$

with

$$B_{mn} \equiv \frac{1}{L}\int_0^L \varphi_m(x)\varphi_n(x)\mathrm{d}x \tag{4}$$

$$v_m \equiv \frac{1}{L}\int_0^L \varphi_m(x)U(x)\mathrm{d}x \tag{5}$$

Show that the matrix **B** is symmetric positive definite. Show that $\bar{U}(x)$ can be expressed as the result of a filtering operation

$$\bar{U}(x) = \int_{x-L}^{x} G(r,x)U(x-r)\mathrm{d}r \tag{6}$$

where the filter is

$$G(r,x)=\frac{1}{L}\sum_{n=1}^{N}\sum_{m=1}^{N}B_{mn}^{-1}\varphi_n(x)\varphi_m(x-r) \tag{7}$$

and B_{mn}^{-1} denotes the *m*-*n* element of the inverse of the matrix **B**. Argue, both from Eq. (6) and from the equation for $\partial\chi/\partial a_m$, that the filtered residual $\bar{u}(x)$ is zero.

Solution

By substituting Eq. (1) into Eq. (2) and differentiating with respect to a_m, we get

$$\begin{aligned}\frac{\partial\chi}{\partial a_m}&=\frac{\partial}{\partial a_m}\left[\frac{1}{L}\int_0^L\left[\sum_{n=1}^{N}a_n\varphi_n(x)-U(x)\right]^2\mathrm{d}x\right]\\&=2\frac{1}{L}\int_0^L\left[\sum_{n=1}^{N}a_n\varphi_n(x)-U(x)\right]\varphi_m(x)\mathrm{d}x\end{aligned} \tag{8}$$

Since a_n is a set of coefficients which minimize Eq. (2), then Eq. (8) should equal zero. This is

$$2\frac{1}{L}\int_0^L\left[\sum_{n=1}^{N}a_n\varphi_n(x)-U(x)\right]\varphi_m(x)\mathrm{d}x=0 \tag{9}$$

After dropping the constant 2 and rearranging the remaining terms we have

$$\frac{1}{L}\int_0^L\left[\varphi_m(x)\sum_{n=1}^{N}a_n\varphi_n(x)\right]\mathrm{d}x=\underbrace{\frac{1}{L}\int_0^L\varphi_m(x)U(x)\mathrm{d}x}_{v_m} \tag{10}$$

The right hand side term is v_m as shown in Eq. (10). As for the left hand side of Eq. (10), we could write

$$\frac{1}{L}\int_0^L\left[\varphi_m(x)\sum_{n=1}^{N}a_n\varphi_n(x)\right]\mathrm{d}x=\sum_{n=1}^{N}a_n\underbrace{\left[\frac{1}{L}\int_0^L\left[\varphi_m(x)\varphi_n(x)\right]\mathrm{d}x\right]}_{B_{mn}} \tag{11}$$

Here Eq. (3) holds.
Based on Eq. (4) we can rewrite matrix **B** into the following form.

$$\mathbf{B}=\frac{1}{L}\int_0^L \boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathrm{T}}\mathrm{d}x \tag{12}$$

where $\boldsymbol{\varphi}$ is a column vector

$$\boldsymbol{\varphi}=\begin{Bmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_N\end{Bmatrix} \tag{13}$$

Let **y** be an arbitrary non-zero vector, do the following matrix multiplication

$$\begin{aligned}
&\mathbf{y}^{\mathrm{T}}\mathbf{B}\mathbf{y}\\
&=\mathbf{y}^{\mathrm{T}}\left(\frac{1}{L}\int_0^L \boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathrm{T}}\mathrm{d}x\right)\mathbf{y}\\
&=\frac{1}{L}\int_0^L \mathbf{y}^{\mathrm{T}}\boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathrm{T}}\mathbf{y}\mathrm{d}x\\
&=\frac{1}{L}\int_0^L \left(\mathbf{y}^{\mathrm{T}}\boldsymbol{\varphi}\right)\left(\boldsymbol{\varphi}^{\mathrm{T}}\mathbf{y}\right)\mathrm{d}x\\
&=\frac{1}{L}\int_0^L \left(\sum_{j=1}^{N} y_j\varphi_j\right)\left(\sum_{j=1}^{N} y_j\varphi_j\right)\mathrm{d}x\\
&=\frac{1}{L}\int_0^L \left(\sum_{j=1}^{N} y_j\varphi_j\right)^2\mathrm{d}x
\end{aligned} \tag{14}$$

Since $\varphi^2 \geq 0$ and φ is a basis function, the integral of φ^2 in [0, L] should not be zero. Therefore for arbitrary non-zero vector **y**, Eq. (14) satisfies

$$\mathbf{y}^{\mathrm{T}}\mathbf{B}\mathbf{y}>0 \tag{15}$$

This is equivalent to that Eq. (12) is a positive definite matrix.
In fact, Eq. (7) can be expressed in matrix notations

$$G(r,x)=\frac{1}{L}\sum_{n=1}^{N}\sum_{m=1}^{N}B_{mn}^{-1}\varphi_n(x)\varphi_m(x-r)=\frac{1}{L}\left(\boldsymbol{\varphi}(x)\right)^{\mathrm{T}}\mathbf{B}^{-1}\boldsymbol{\varphi}(x-r) \tag{16}$$

where $\mathbf{B}^{-1}$ is the inverse matrix of **B**. Noting that **B** and $\mathbf{B}^{-1}$ only depend on x, substitute Eq. (16) into the right hand side of Eq. (6) we obtain

$$\begin{aligned}&\int_{x-L}^{x}\left[\frac{1}{L}\left(\boldsymbol{\varphi}(x)\right)^{\mathrm{T}}\mathbf{B}^{-1}\boldsymbol{\varphi}(x-r)\right]U(x-r)\mathrm{d}r\\&=\left(\boldsymbol{\varphi}(x)\right)^{\mathrm{T}}\mathbf{B}^{-1}\frac{1}{L}\int_{x-L}^{x}\boldsymbol{\varphi}(x-r)U(x-r)\mathrm{d}r\end{aligned}\tag{17}$$

Let’s take out the integral term in Eq. (17) and perform a variable substitution

$$\frac{1}{L}\int_{x-L}^{x}\boldsymbol{\varphi}(x-r)U(x-r)\mathrm{d}r\overset{t=x-r}{=}-\frac{1}{L}\int_{L}^{0}\boldsymbol{\varphi}(t)U(t)\mathrm{d}t=\frac{1}{L}\int_{0}^{L}\boldsymbol{\varphi}(t)U(t)\mathrm{d}t\tag{18}$$

Recalling Eq. (3) and Eq. (5) we can turn Eq. (18) into

$$\frac{1}{L}\int_{0}^{L}\boldsymbol{\varphi}(t)U(t)\mathrm{d}t=\mathbf{Ba}\tag{19}$$

where **a** is a column vector expressed as Eq. (20)

$$\mathbf{a}=\begin{Bmatrix}a_1\\a_2\\\vdots\\a_N\end{Bmatrix}\tag{20}$$

Substitute Eq. (19) back into Eq. (17)

$$\begin{aligned}&\left(\boldsymbol{\varphi}(x)\right)^{\mathrm{T}}\mathbf{B}^{-1}\frac{1}{L}\int_{x-L}^{x}\boldsymbol{\varphi}(x-r)U(x-r)\mathrm{d}r\\&=\left(\boldsymbol{\varphi}(x)\right)^{\mathrm{T}}\mathbf{B}^{-1}\mathbf{Ba}\\&=\left(\boldsymbol{\varphi}(x)\right)^{\mathrm{T}}\mathbf{a}\\&=\sum_{j=1}^{N}a_j\varphi_j\\&=\bar{U}(x)\end{aligned}\tag{21}$$

It means that Eq. (6) holds true.