Solution to Ex. 6.25

of *Turbulent Flows* by Stephen B. Pope, 2000

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Show that application of the incompressibility condition $\kappa_i \Phi_{ij}(\boldsymbol{\kappa}) = 0$ to Eq. (6.192) yields

$$B(\kappa) = -\frac{A(\kappa)}{\kappa^2} \tag{1}$$

Using the results of Exercise (6.24), show that the energy-spectrum function corresponding to Eq. (6.192) is

$$E(\kappa) = 6\pi\kappa^2 A(\kappa) + 2\pi\kappa^4 B(\kappa) \tag{2}$$

Hence deduce Eq. (6.193).

Solution

Multiply κ_i on both side of Eq. (6.192), we have

$$\kappa_i \Phi_{ij}(\boldsymbol{\kappa}) = \kappa_i A(\kappa)\delta_{ij} + \kappa_i B(\kappa)\kappa_i\kappa_j = 0 \tag{3}$$

Thus

$$0 = \kappa_i A(\kappa)\delta_{ij} + \kappa_i B(\kappa)\kappa_i\kappa_j = \kappa_j A(\kappa) + \kappa^2 B(\kappa)\kappa_j \tag{4}$$

If Eq. (4) holds for any κ_j. Then

$$A(\kappa) + \kappa^2 B(\kappa) = 0 \tag{5}$$

This leads to

$$B(\kappa) = -\frac{A(\kappa)}{\kappa^2} \tag{6}$$

The energy-spectrum function is

$$
\begin{aligned}
E(\kappa) &= \oint \frac{1}{2}\Phi_{ii}(\boldsymbol{\kappa})\,\mathrm{d}S(\kappa) \\
&= \oint \frac{1}{2}\left(3A(\kappa) + B(\kappa)\kappa_i\kappa_i\right)\mathrm{d}S(\kappa) \\
&= \frac{3}{2}A(\kappa)\oint \mathrm{d}S(\kappa) + \frac{1}{2}B(\kappa)\oint \kappa_i\kappa_i\,\mathrm{d}S(\kappa) \\
&= 6\pi\kappa^2 A(\kappa) + \frac{1}{2}B(\kappa)\times 3\times\frac{4}{3}\pi\kappa^4 \\
&= 6\pi\kappa^2 A(\kappa) + 2\pi\kappa^4 B(\kappa)
\end{aligned} \tag{7}
$$

Substitute Eq. (6) into Eq. (7), we obtain

$$
A(\kappa) = \frac{E(\kappa)}{4\pi\kappa^2} \tag{8}
$$

Substituting Eq. (8) and Eq. (6) into Eq. (6.192), we could obtain

$$
\Phi_{ij}(\boldsymbol{\kappa}) = \frac{E(\kappa)}{4\pi\kappa^2}\delta_{ij} - \frac{E(\kappa)}{4\pi\kappa^2}\frac{\kappa_i\kappa_j}{\kappa^2} = \frac{E(\kappa)}{4\pi\kappa^2}\left(\delta_{ij} - \frac{\kappa_i\kappa_j}{\kappa^2}\right) = \frac{E(\kappa)}{4\pi\kappa^2}P_{ij}(\boldsymbol{\kappa}) \tag{9}
$$