# Solution to Ex. 7.11

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In non-swirling statistically axisymmetric flows, the Reynolds equations of the pipe flow in polar-cylindrical coordinates are:
continuity equation

$$\frac{\partial \langle U \rangle}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r\langle V \rangle) = 0 \tag{1}$$

axial momentum equation

$$\frac{\bar{\mathrm{D}}\langle U \rangle}{\bar{\mathrm{D}}t} = -\frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x} - \frac{\partial}{\partial x}\langle u^2 \rangle - \frac{1}{r}\frac{\partial}{\partial r}(r\langle uv \rangle) + \nu \nabla^2 \langle U \rangle \tag{2}$$

and radial momentum equation

$$\frac{\bar{\mathrm{D}}\langle V \rangle}{\bar{\mathrm{D}}t} = -\frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial r} - \frac{\partial}{\partial x}\langle uv \rangle - \frac{1}{r}\frac{\partial}{\partial r}(r\langle v^2 \rangle) + \frac{\langle w^2 \rangle}{r} + \nu\left(\nabla^2 \langle V \rangle - \frac{\langle V \rangle}{r^2}\right) \tag{3}$$

where

$$\frac{\bar{\mathrm{D}}}{\bar{\mathrm{D}}t} = \frac{\partial}{\partial t} + \langle U \rangle \frac{\partial}{\partial x} + \langle V \rangle \frac{\partial}{\partial r} + \frac{\langle W \rangle}{r}\frac{\partial}{\partial \theta} \tag{4}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{5}$$

Consider the fully developed turbulent pipe flow. The flow is statistically stationary and the statistics are only depend on r-coordinate. Then, the continuity equation can be rewritten as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\langle V \rangle) = 0 \tag{6}$$

$r$ is in range $(0, R)$, here we can assume that

$$\frac{\partial}{\partial r}(r\langle V \rangle) = 0 \tag{7}$$

with the assumption that

$$\lim_{r \to 0} \frac{\langle V \rangle}{r} = 0 \tag{8}$$

integrate Eq. (7) with respect to $r$, we get

$$r\langle V\rangle = C_{\rm v1} \tag{9}$$

where $C_{\rm v1}$ is constant. Apply the boundary condition that $\langle V\rangle = 0$ at $r = R$, then $C_{\rm v1} = 0$. And consequently, $\langle V\rangle = 0$. Hence, the radial momentum equation can be rewritten as

$$0 = -\frac{1}{\rho}\frac{\partial\langle p\rangle}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}(r\langle v^2\rangle) + \frac{\langle w^2\rangle}{r} \tag{10}$$

integrate with respect to $r$, we get

$$\frac{\langle p\rangle}{\rho} + \langle v^2\rangle = \int \frac{\langle w^2\rangle - \langle v^2\rangle}{r}\,{\rm d}r + C_{\rm v2} \tag{11}$$

where $C_{v2}$ is constant. With boundary condition $\langle v^2\rangle = \langle w^2\rangle = 0$ and $\langle p\rangle = p_{\rm w}$ at $r = R$, we have

$$\frac{p_{\rm w}}{\rho} = C_{\rm v2} \tag{12}$$

then

$$\frac{\langle p\rangle}{\rho} + \langle v^2\rangle = \int \frac{\langle w^2\rangle - \langle v^2\rangle}{r}\,{\rm d}r + \frac{p_{\rm w}}{\rho} \tag{13}$$

note that the statistics of fluctuating velocity are independent on $x$, we have

$$\frac{\partial\langle p\rangle}{\partial x} = \frac{{\rm d}p_{\rm w}}{{\rm d}x} \tag{14}$$

substitute Eq. (14) into Eq. (2)

$$0 = -\frac{1}{\rho}\frac{{\rm d}p_{\rm w}}{{\rm d}x} - \frac{1}{r}\frac{\partial}{\partial r}(r\langle uv\rangle) + \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\langle U\rangle}{\partial r}\right) \tag{15}$$

the shear stress is defined as

$$\tau \equiv \rho\nu\frac{{\rm d}\langle U\rangle}{{\rm d}r} - \rho\langle uv\rangle \tag{16}$$

rearrange Eq. (15) and make use of Eq. (16)

$$0 = -\frac{{\rm d}p_{\rm w}}{{\rm d}x} + \frac{1}{r}\frac{\partial}{\partial r}(r\tau) \tag{17}$$

rearrange

$$r\frac{{\rm d}p_{\rm w}}{{\rm d}x} = \frac{\partial}{\partial r}(r\tau) \tag{18}$$

integrate Eq. (18) with respect to $r$

$$\begin{aligned} r\tau &= \frac{1}{2}r^2\frac{{\rm d}p_{\rm w}}{{\rm d}x} + C_{\rm v3} \\ \tau &= \frac{1}{2}r\frac{{\rm d}p_{\rm w}}{{\rm d}x} + C'_{\rm v3} \end{aligned} \tag{19}$$

Since the flow in the pipe is fully developed, it is reasonable to assume that the flow is axisymmetric and the shear stress along the central line ($r = 0$) is 0. Then $C'_{\mathrm{v}3} = 0$. And finally

$$\tau = \frac{1}{2} r \frac{\mathrm{d}p_{\mathrm{w}}}{\mathrm{d}x} \tag{20}$$

If we introduce the wall shear stress

$$\tau_{\mathrm{w}} = -\tau(R) \tag{21}$$

thus

$$-\tau_{\mathrm{w}} = \frac{1}{2} R \frac{\mathrm{d}p_{\mathrm{w}}}{\mathrm{d}x} \tag{22}$$

after rearrange the terms, we get

$$-\frac{\mathrm{d}p_{\mathrm{w}}}{\mathrm{d}x} = 2\frac{\tau_{\mathrm{w}}}{R} \tag{23}$$